Inequality (**Mihalcea Andrei Stefan**)
https://www.linkedin.com/groups/8313943/8313943-6379739547886063618

Let $a,b,c>0$. Show that $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}\ge\dfrac{3\sqrt[3]{5+\sum\frac{ab}{c^2}}}{2}$.

Solution by Arkady Alt , **San Jose**, **California**, **USA**.

We will prove more general, namely, that for any real $k>1$ holds inequality:

(**1**) $\quad \dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}\ge 3\sqrt[3]{\dfrac{k+\sum\frac{ab}{c^2}}{k+3}}$

Let $x:=\dfrac{b}{c},y:=\dfrac{c}{a},z:=\dfrac{a}{b}$. Then $xyz=1,\ \sum\dfrac{a}{b}=x+y+z,\sum\dfrac{ab}{c^2}=\sum\dfrac{x}{y}=$

$\sum zx^2$ and, therefore, inequality (**1**) becomes $x+y+z\ge 3\sqrt[3]{\dfrac{k+\sum\frac{ab}{c^2}}{k+3}}\iff$

$k+\sum zx^2\le\dfrac{(k+3)(x+y+z)^3}{27}$ or, in homogeneous form:

(**1**) $\quad xy^2+yz^2+zx^2+kxyz\le\dfrac{(k+3)(x+y+z)^3}{27}$.

Since* $xy^2+yz^2+zx^2+xyz\le\dfrac{4(x+y+z)^3}{27}$ and $xyz\le\dfrac{(x+y+z)^3}{27}$ then

$xy^2+yz^2+zx^2+kxyz=xy^2+yz^2+zx^2+xyz+(k-1)xyz\le$

$\dfrac{4(x+y+z)^3}{27}+\dfrac{(k-1)(x+y+z)^3}{27}=\dfrac{(k+3)(x+y+z)^3}{27}$.

* it is well known cyclic inequality of V. Cirtoaje (see Problem16, p.7 in the book V.Cirtoage "Algebraic inequalities.Old and Methods").

I quote his short proof:

Let $c=\min\{a,b,c\}$ and $x:=a-c\ge 0,y:=b-c\ge 0$ then $a=x+c,b=y+c$ and

$4(a+b+c)^3-27(abc+a^2b+b^2c+c^2a)=$

$4(3c+x+y)^3-27\big((x+c)(y+c)c+(x+c)^2(y+c)+(y+c)^2c+c^2(x+c)\big)=$

$4x^3-15x^2y+12xy^2+4y^3+9cx^2-9cxy+9cy^2=(4x+y)(x-2y)^2+9c(x^2-xy+y^2)\ge 0.$